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Weak Tchebyshev-Spaces and Generalizations of a Theorem by Krein

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1. INTRODUCTION

Let U be an n-dimensional space of real valued functions defined on some totally ordered set M. We call U a weak Tchebyshev-space, if no $f \in U$ has more than $n - 1$ changes of sign. The most important example of a weak Tchebyshev-space is given by the polynomial splines with fixed knots (see [2]). If, in addition, no function $f \neq 0$ in U has more than $n - 1$ zeros, i.e., if U is a Tchebyshev-space as well, U is called an oriented Tchebyshev-space. A well known theorem of Krein (see [3]) states that every n-dimensional Tchebyshev-space of continuous functions on an open interval contains an $(n - 1)$-dimensional Tchebyshev-space. This result was generalized by Zielke [10] for the case of oriented Tchebyshev-spaces. He supposed (1) that the domain of definition M has no smallest and no greatest element and (2) that between any two points of M there is another point of M. Recently Zalik [8] has shown that the result of Zielke is still valid even without the second assumption (2). On the other hand, every weak Tchebyshev-space of dimension n contains a weak Tchebyshev-space of dimension $n - 1$ without any restriction on M (see [6]). It is the purpose of this paper to derive the result of Zalik from this theorem on weak Tchebyshev-spaces. The basic tool is a characterization of a weak Tchebyshev-space U by means of the generalized Vandermonde-determinant. This characterization was shown by Jones and Karlovitz [1] where U consists of continuous functions on a real interval. We generalize their result to the case of weak Tchebyshev-spaces of (not necessarily continuous) functions on arbitrary totally ordered sets. As an application we consider finally the question, if there exist nonnegative functions and positive functions in a weak Tchebyshev-space and in an oriented Tchebyshev-space, respectively.

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2. Definitions and Basic Properties

The notion of alternation is one possible basis for the concept of weak and oriented Tchebyshev-spaces (see e.g. [6, 7, 10]).

Definition 1. Let M be a totally ordered set and $f \in \mathbb{R}^M$, the space of all mappings from M to $\mathbb{R}$. We call n points $x_1 < \cdots < x_n$ from M an alternation of f of length n, iff

$$f(x_i) \cdot f(x_{i+1}) < 0, \qquad i = 1,..., n-1.$$

We can now state the definition of weak and oriented Tchebyshev-spaces.

Definition 2. Let M be a totally ordered set and U an n-dimensional subspace of $\mathbb{R}^M$. U is called a weak Tchebyshev-space (or weak T-space for short), iff no $f \in U$ has an alternation of length $n+1$. If, in addition, U is a Tchebyshev-space (T-space), i.e., if no $f \neq 0$ from U has more than $n-1$ zeros, we call U an oriented Tchebyshev-space (oriented T-space).

Another characterization of weak and oriented T-spaces comes from the generalized Vandermonde-determinant

$$\det\begin{pmatrix} f_1, ..., f_n \\ x_1, ..., x_n \end{pmatrix} = \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}.$$

The following theorem goes back to Zielke [10].

Theorem 1. *Let M be totally ordered and U be an n-dimensional subspace of $\mathbb{R}^M$. Then the following assertions are equivalent:*

(1) *U is an oriented T-space.*

(2) *There is a basis $f_1, ..., f_n$ of U with*

$$\det\begin{pmatrix} f_1, ..., f_n \\ x_1, ..., x_n \end{pmatrix} > 0$$

for arbitrary $x_1 < \cdots < x_n$ in M.

This theorem is an immediate consequence of Theorem 2 below. An analogous characterization of weak T-spaces in the case of continuous functions on a real interval was proved by Jones–Karlovitz [1]. We now show that their assertion is still valid in the more general situation considered here and therefore our definition of weak T-spaces is in agreement with [2] and [8]. The proof is based on the following lemma.

LEMMA 1. *Let M be totally ordered, U an n-dimensional weak T-space in $\mathbb{R}^M$, $f_1,\ldots,f_n$ a basis of U and let $z_1 < \cdots < z_{n-1}$ be points in M such that there are $x_1,\ldots,x_n$ in M with*

$$\{z_1,\ldots,z_{n-1}\} \subset \{x_1,\ldots,x_n\}$$

and

$$\det\begin{pmatrix} f_1,\ldots,f_n \\ x_1,\ldots,x_n \end{pmatrix} \neq 0.$$

Let $f \in U$ with $f(z_i) = 0$, $i = 1,\ldots,n-1$. Then for arbitrary y_i, $\bar{y}_j$ with

$$\begin{matrix} z_{i-1} < y_i < z_i \\ z_{j-1} < \bar{y}_j < z_j \end{matrix} \qquad (1 \leqslant i \leqslant j \leqslant n,\ y_i \leqslant \bar{y}_j)$$

the inequality

$$(-1)^{i+j} \cdot f(y_i) \cdot f(\bar{y}_j) \geqslant 0$$

holds. Here we set $z_0 := -\infty$ and $z_n := +\infty$.

Proof. Let us assume the contrary: we have y_i, $\bar{y}_j$ as above with

$$(-1)^{i+j} \cdot f(y_i) \cdot f(\bar{y}_j) < 0.$$

Because

$$\det\begin{pmatrix} f_1,\ldots,f_n \\ x_1,\ldots,x_n \end{pmatrix} \neq 0$$

there is an $h \in U$ with

$$h(z_k) = (-1)^{k+i+r} \cdot \operatorname{sign} f(y_i), \qquad 1 \leqslant k \leqslant n-1,$$

where $r = 1$ in the case $i \leqslant k \leqslant j-1$ and $r = 0$ otherwise. Then the function $f + \lambda h$ has for sufficiently small $\lambda > 0$ an alternation of length $n+1$ in the increasingly ordered points $z_1,\ldots,z_{n-1}, y_i, \bar{y}_j$. This contradiction ends the proof. ∎

Note that in the case $n = 1$ the lemma simply states that no $f \in U$ has an alternation of length 2.

Lemma 1 is a generalization of the well known fact that a function from an n-dimensional oriented T-space with $n-1$ zeros has constant sign between these zeros and changes sign at each of them (see [10]). We can now prove a result for weak T-spaces which is analogous to Theorem 1.

THEOREM 2. *Let M be totally ordered and U be an n-dimensional subspace of $\mathbb{R}^M$. Then the following assertions are equivalent:*

(1) *U is a weak T-space.*

(2) *There is a basis $f_1,\dots,f_n$ of U with*

$$\det\begin{pmatrix} f_1,\dots,f_n \\ x_1,\dots,x_n \end{pmatrix} \geqslant 0$$

for arbitrary $x_1 < \cdots < x_n$ in M.

Proof. (1) $\Rightarrow$ (2). Let $f_1,\dots,f_n$ be a basis of U and let $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ be points in M with

$$\det\begin{pmatrix} f_1,\dots,f_n \\ x_1,\dots,x_n \end{pmatrix} \neq 0 \neq \det\begin{pmatrix} f_1,\dots,f_n \\ y_1,\dots,y_n \end{pmatrix}.$$

We consider the functions g_k, $1 \leqslant k \leqslant n$, with

$$g_k(x) := \left(\det\begin{pmatrix} f_1,\dots,f_n \\ x_1,\dots,x_n \end{pmatrix}\right)^{-1} \cdot \det\begin{pmatrix} f_1,\dots,f_n \\ x, x_1,\dots,x_{k-1}, x_{k+1},\dots,x_n \end{pmatrix}.$$

Let there exist a j_0 with $x_{j_0} \notin \{y_1,\dots,y_n\}$. Since the g_k span U as well, there is a y_m with $g_{j_0}(y_m) \neq 0$. Clearly $y_m \notin \{x_1,\dots,x_n\}$. If we now set

$$\begin{aligned} z_i &:= x_i \qquad \text{for} \quad i = 1,\dots,j_0 - 1, \\ z_i &:= x_{i+1} \qquad \text{for} \quad i = j_0,\dots,n-1, \\ z_0 &:= -\infty, \qquad z_n := +\infty, \end{aligned}$$

and if we set $w_1 < \cdots < w_n$ with

$$\{w_i \mid i = 1,\dots,n\} = \{z_i \mid i = 1,\dots,n-1\} \cup \{y_m\},$$

from the relation $z_{i-1} < y_m < z_i$ it follows that

$$\det\begin{pmatrix} f_1,\dots,f_n \\ w_1,\dots,w_n \end{pmatrix} = (-1)^{i+1} \cdot g_{j_0}(y_m) \cdot \det\begin{pmatrix} f_1,\dots,f_n \\ x_1,\dots,x_n \end{pmatrix}.$$

From Lemma 1 and from the equality

$$(-1)^{j_0+1} \cdot g_{j_0}(x_{j_0}) = 1$$

we see that

$$(-1)^{i+1} \cdot g_{j_0}(y_m) > 0.$$

Therefore the two determinants

$$\det\begin{pmatrix} f_1,\dots,f_n \\ x_1,\dots,x_n \end{pmatrix}, \qquad \det\begin{pmatrix} f_1,\dots,f_n \\ w_1,\dots,w_n \end{pmatrix}$$

are of the same sign ($\neq 0$). Repeated application of this argument shows that (1) implies (2).

(2) $\Rightarrow$ (1). This implication will not be needed in the following; we therefore refer to [12] for a proof. ∎

Theorem 1 now follows immediately from Theorem 2. It is well known that an n-dimensional subspace U of $\mathbb{R}^M$ is a T-space if and only if there is a basis $f_1, ..., f_n$ of U with

$$\det \begin{pmatrix} f_1, ..., f_n \\ x_1, ..., x_n \end{pmatrix} \neq 0$$

for arbitrary $x_1, ..., x_n$ in M (see e.g. [10]).

3. Subspaces of Weak and Oriented Tchebyshev-Spaces

In an earlier paper [6] we have shown the following theorem, which was proved independently from us and with a different method of proof in the special case of continuous functions on a compact real interval by Sommer and Strauss [5].

THEOREM 3. *Let M be totally ordered and $U \subset \mathbb{R}^M$ a weak T-space of dimension $n \geqslant 2$. Then there is a weak T-space V of dimension $n - 1$ with $V \subset U$.*

We now derive from this theorem the following result due to Zalik [8], which is an improvement of a theorem of Zielke ([10], see also [6]).

THEOREM 4. *Let M be totally ordered without greatest and smallest element and let $U \subset \mathbb{R}^M$ be an oriented T-space of dimension $n \geqslant 2$. Then there is an oriented T-space V of dimension $n - 1$ with $V \subset U$.*

Proof. By Theorem 3 there is a weak T-space V of dimension $n - 1$ with $V \subset U$. We will show that V is already an oriented T-space. To do this, let $f_1, ..., f_n$ be a basis of U with the following properties: (1) $f_1, ..., f_{n-1}$ is a basis of V with

$$\det \begin{pmatrix} f_1, ..., f_{n-1} \\ x_1, ..., x_{n-1} \end{pmatrix} \geqslant 0$$

for arbitrary $x_1 < \cdots < x_{n-1}$ (see Theorem 2), (2) for arbitrary $x_1 < \cdots < x_n$ we have

$$\det \begin{pmatrix} f_1, ..., f_n \\ x_1, ..., x_n \end{pmatrix} > 0$$

(see Theorem 1). Let us assume there are $y_1 < \cdots < y_{n-1}$ in M with

$$\det\begin{pmatrix} f_1, \ldots, f_{n-1} \\ y_1, \ldots, y_{n-1} \end{pmatrix} = 0.$$

We choose $y_n > y_{n-1}$ and conclude from

$$\det\begin{pmatrix} f_1, \ldots, f_n \\ y_1, \ldots, y_n \end{pmatrix} > 0$$

that there is an i_0 with

$$\det\begin{pmatrix} f_1, \ldots, f_{n-1} \\ y_1, \ldots, y_{i_0-1}, y_{i_0+1}, \ldots, y_n \end{pmatrix} > 0.$$

Let

$$f(x) := \det\begin{pmatrix} f_1, \ldots, f_n \\ y_1, \ldots, y_{i_0-1}, y_{i_0+1}, \ldots, y_n, x \end{pmatrix}.$$

Then $f \in U$ and $f_1, \ldots, f_{n-1}, f$ is another basis of U with property (2). Let $y_0 < y_1$. If we now use the relations

$$f(y_j) = 0 \qquad \text{for} \quad j = 1, \ldots, n-1, \quad j \neq i_0,$$

$$(-1)^{n-i_0} \cdot f(y_{i_0}) > 0,$$

and our assumption and expand by the last column we get

$$0 < \det\begin{pmatrix} f_1, \ldots, f_{n-1}, f \\ y_0, \ldots, y_{n-1} \end{pmatrix}$$

$$= (-1)^{n+i_0+1} \cdot f(y_{i_0}) \cdot \det\begin{pmatrix} f_1, \ldots, f_{n-1} \\ y_0, \ldots, y_{i_0-1}, y_{i_0+1}, \ldots, y_{n-1} \end{pmatrix}.$$

But since the right side of the last equation is not positive, we have a contradiction. ▮

The proof of Theorem 4 shows that *any* $(n-1)$-dimensional subspace of an n-dimensional oriented T-space is again an oriented T-space, provided it is a weak T-space and the domain of definition has no greatest and no smallest element.

4. Remarks and Applications

If M is a real interval and U an n-dimensional vectorspace of continuous functions on M, then U is an oriented T-space if and only if U is a T-space. Thus Theorem 4 is a generalization of the result by Krein (see [3]) mentioned above. Repeated application of Theorem 3 shows that every weak T-space of dimension n has a basis $f_1, ..., f_n$ such that, for any k with $1 \leqslant k \leqslant n$, the functions $f_1, ..., f_k$ span a weak T-space. Similarly, if the domain of definition has no smallest and no greatest element, from Theorem 4 there follows the analogous result for oriented T-spaces. From these remarks one can easily deduce the following theorem on the existence of nonnegative and positive functions in weak and oriented T-spaces.

THEOREM 5. *Let M be totally ordered and let $U \subset \mathbb{R}^M$ be a weak T-space. Then there is an $f \neq 0$ in U with $f \geqslant 0$. If M has no smallest and no greatest element and if U is an oriented T-space in $\mathbb{R}^M$, there is an f in U with $f > 0$.*

There is a further application in this direction. The following result can also be derived from the Tchebyshev Equioscillation Theorem (see [4]). But when it is shown with interpolation methods only, it can serve as a starting point for a proof of the theorem by Tchebyshev.

THEOREM 6. *Let U be a T-space in $C[a, b]$. Then there is an $f \in U$ with $f > 0$.*

Proof. By Theorem 5 there is a nontrivial f in U with $f \geqslant 0$. If $x_1, ..., x_r$ are the zeros of f, then $r \leqslant n - 1$ and we can choose $g \in U$ with $g(x_i) = 1$, $i = 1, ..., r$. Then, for sufficiently great $\lambda > 0$ we have $\lambda f + g > 0$ on $[a, b]$. ∎

Theorem 6 is not valid in the case of halfopen intervals as shown by the simple example $U_0 = \text{span}\{\sin, \cos\}$ in $C[0, \pi[$. It should be noted that, in general, there is no basis $f_1, ..., f_n$ of U with the property that $f_1, ..., f_k$ span a T-space for $1 \leqslant k \leqslant n$ if U is a T-space in $C[a, b[$ or $C[a, b]$. For various examples in this connection see [11]. We also refer to a paper of Zielke [9] which contains a result on subspaces of periodic Tchebyshev-spaces.

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